# Electrical Resistances in Products of Graphs

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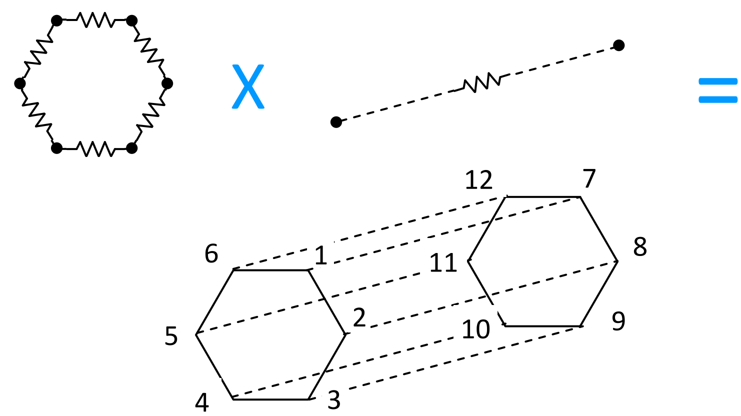
# Contents

# I. Introduction

The analysis of the behavior of voltages and currents in complex circuits can be simplified through the study and understanding of behaviors of more elementary circuits. In this paper we will examine the use of graph theory to determine the behavior of voltages and current in resistor networks. The main results are based on the theorems and proofs presented in "Random Walks and Electrical Resistances in Products of Graphs" by Béla Bollobás and Graham Brightwell [[1]].

One elementary circuit we will consider is the product of one resistor and six resistors connected in a hexagon. This circuit can be represented by the product graph of a six-vertex cycle and a two-vertex path, $C_6 \times P_2$.




If the edges of the product graph are unit resistors, questions we might ask, for example, are: What is the effective resistance between vertices 1 and 4? How does that compare with the effective resistance between vertices 1 and 10? While some of the answers may seem intuitive, others, as we will see, are more unexpected.

Chapter II contains definitions and properties related to electrical circuits and explains how they can be used to determine effective resistances in a resistor network. Chapter III covers graph theory definitions and concepts. In Chapter IV, we present the main results and illustrative examples. In Chapter V, we prove the results.

# II. Basic Electrical Properties

The electrical properties described in this chapter are some of the more basic elements of electrical theory. The interested reader can find additional information in “Electric Circuits” by James W. Nilsson [**Error! Reference source not found.**] and “Random Walks and Electrical Resistances” by Peter G. Doyle and J. Laurie Snell [[2]].

## A. Electrical Terminology

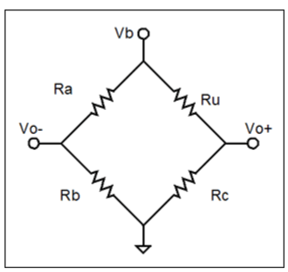
Two basic electrical elements are resistors and resistor networks.

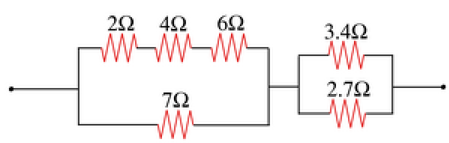
### 1. Resistor

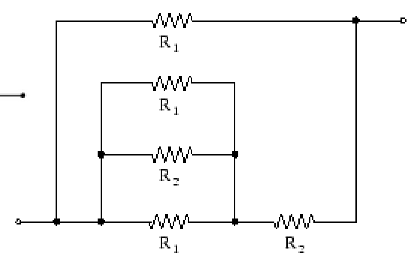
A resistor is a device in an electric circuit with two terminals that impedes current flow. It is represented in circuit diagrams by a zig-zag line with its resistance value R.


### 2. Resistor Network

A resistor network is an electrical circuit consisting of a set of connected resistors. Any point where two or more terminals meet is called a node. Here are some typical examples:



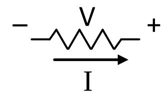
## B. Electrical Laws

Three of the most fundamental relationships in electricity are described by Ohm's law and Kirchoff's current and voltage laws. These rules are used to determine energy supply and dissipation in a resistor network.

### 1. Ohm's Law

Ohm's Law describes the relationship between the voltage drop V across a resistor R and the current I flowing through the resistor.

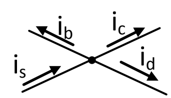
$$V = IR$$




### 2. Kirchoff's Current Law

Kirchoff's Current Law is a description of how current is distributed at the node in an electrical circuit. It states that the algebraic sum of all the currents at any node in a circuit equals zero, or equivalently, that the current flowing out of a node is equal to the current flowing into it.

$$\sum i_{in} = \sum i_{out}$$

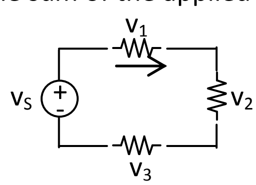



$$i_s = i_b + i_c + i_d$$

### 3. Kirchoff's Voltage Law

Kirchoff's Voltage Law is a description of how voltage is distributed within a closed path of an electrical circuit. It states that the algebraic sum of all the voltages around any closed path in a circuit equals zero, or equivalently, that the sum of voltage drops around a closed path within a circuit is equal to the sum of the applied voltages.

$$\sum_s v_s = \sum_j v_j$$

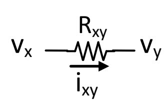



$$v_s = v_1 + v_2 + v_3$$

### 4. Conservation of Energy: Dissipation and Supply

A voltage applied between nodes a and b in a resistor network establishes voltages $v_x$ and $v_y$ at the terminals of each resistor $R_{xy}$ and a current $i_{xy}$ flowing through the resistor. The energy dissipated by the resistor is

$$i_{xy}{}^2 R_{xy},$$




By Ohm's Law, the energy dissipated by a resistor of unit resistance, $R_{xy} = 1$, is equivalent to

$$i_{xy}(v_x - v_y) = \frac{(v_x - v_y)}{R}(v_x - v_y) = (v_x - v_y)^2.$$

Thus, the total energy dissipation in a resistor network is $E_d = ½ \sum_{Rxy} (v_x - v_y)^2$. We multiply by ½ since the energy dissipated by each resistor is counted twice as $R_{xy}$ and as $R_{yx}$.

If we apply a voltage from a source such as a battery that establishes voltages $v_a$ and $v_b$ at nodes a and b in the network the energy supplied to the network is $E_s = (v_a - v_b)i_a$, where $i_a = \sum_x i_{ax}$.

By the conservation of energy, the energy supplied and the energy dissipated must be equal. So, if we let $v_b = 0$, then we have

$$v_a i_a = (v_a - v_b) i_a = E_s = E_d = \tfrac{1}{2} \sum_{Rxy} (v_x - v_y)^2 = \tfrac{1}{2} \sum_{Rxy} i_{xy}^2 R_{xy}.$$

Refer to Doyle & Snell [[2], p. 61] for a proof of the conservation of energy.

## C. <u>Circuit Analysis</u>

Ohm's law and Kirchoff's current and voltage laws are essential in analysis of electrical circuits. For example, we can use them to analyze effective resistance, transformations, and energy of an electrical circuit.

### 1. Effective Resistance

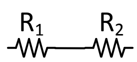
<u>Effective resistance</u> is the voltage drop across a circuit divided by the total current through the circuit.

$$R_{eff} = (v_a - v_b)/i_a$$

For some elementary resistor networks the effective resistance is the equivalent resistance of the value of a single resistor that can be used in place of the resistors in the network.

**a) Resistors in Series**

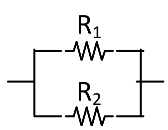
The resistance of two resistors in series is equivalent to the sum of their resistances.




$$R_{eff} = R_1 + R_2$$

**b) Resistors in Parallel**

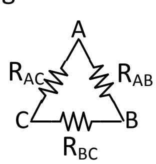
The resistance of two resistors in parallel is equivalent to the product of their resistances divided by the sum of their resistances.

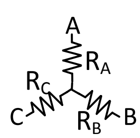



$$R_{eff} = \frac{R_1 R_2}{R_1 + R_2}$$

### 2. Delta-to-Star Transformation

Another type of equivalent resistance is the transformation of three resistors in a <u>delta-network</u> (connected to form a triangle) to a <u>star-network</u> (connected to form a Y) by the following formulas.




$$R_A = \frac{R_{AB} R_{AC}}{R_{AB} + R_{AC} + R_{BC}}$$

$$R_B = \frac{R_{AB} R_{BC}}{R_{AB} + R_{AC} + R_{BC}}$$

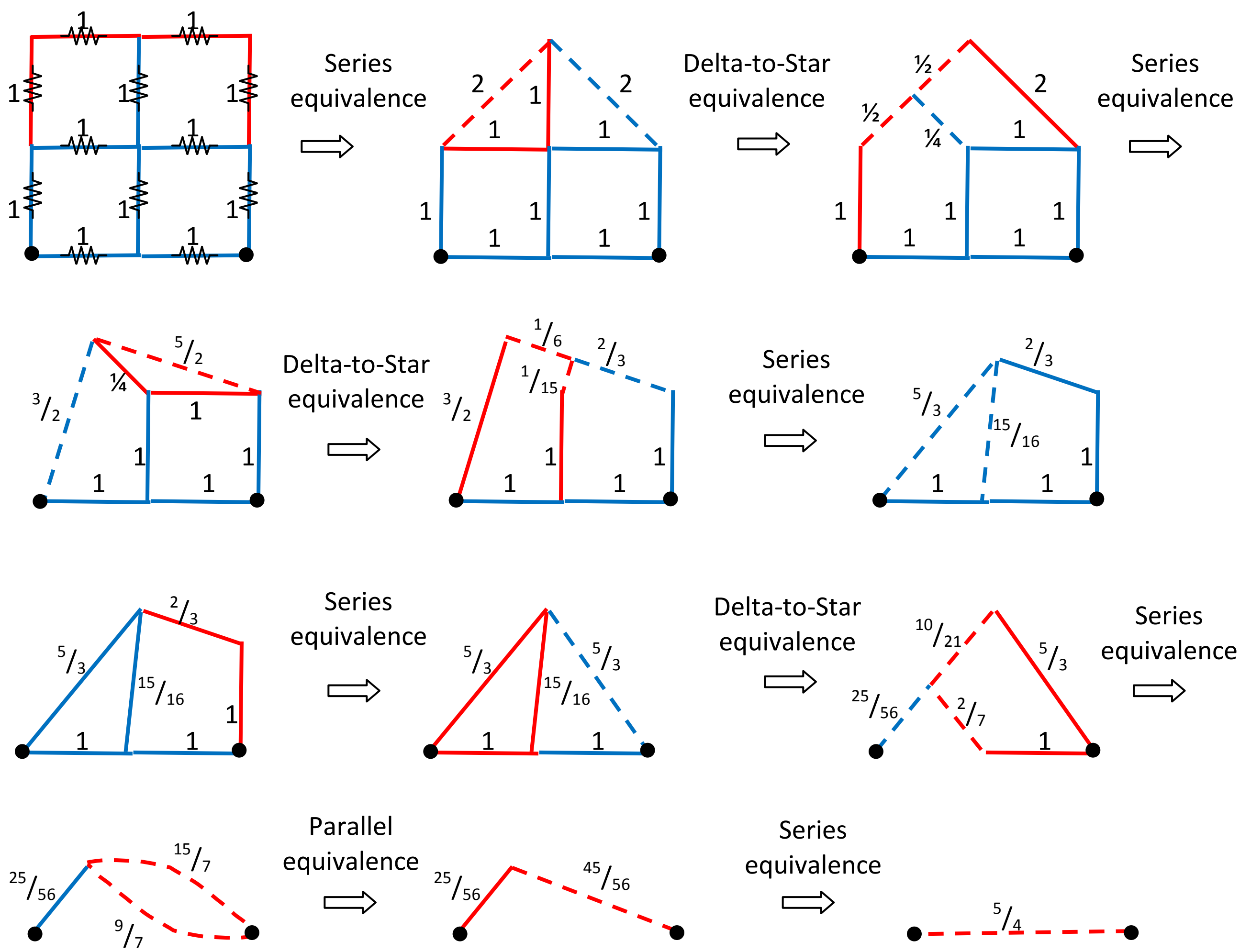
$$R_C = \frac{R_{AC} R_{BC}}{R_{AB} + R_{AC} + R_{BC}}$$

### 3. Effective Resistance Application

One application of equivalent resistance circuits is to determine the effective resistance between two nodes of a resistor network.

For example, suppose we are given a network of resistors of unit resistance connected as shown in the first diagram below and we want to know the effective resistance between the nodes marked with black dots. We can replace resistances in series, resistances in parallel, and delta networks step by step until we are left with one resistance between the two nodes. The value of this resistance is the effective resistance between the two nodes.

To simplify the diagrams, we leave out the resistor symbols. In each step below, we replace the resistances shown in red in the first diagram with their equivalent resistances, which are shown in the next diagram as dashed lined along with their equivalent resistance values.



## 4. Minimum Energy Dissipation

To discuss the minimum energy dissipated in a circuit, we need to define a few new terms and describe a principle of electrical circuits.

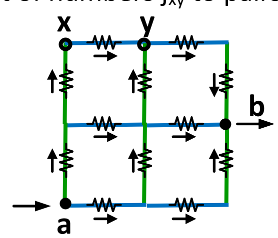
### a) Flow

If we apply a voltage across nodes labeled a and b in a resistor network so that all current enters at a and exits at b and we let x and y be any pair nodes in the circuit, then a <u>flow</u> j from a to b is defined as an assignment of numbers $j_{xy}$ to pairs xy such that

(i) $j_{xy} = -j_{yx}$,

(ii) $\Sigma_x j_{xy} = 0$ if $x \neq$ a, b, and

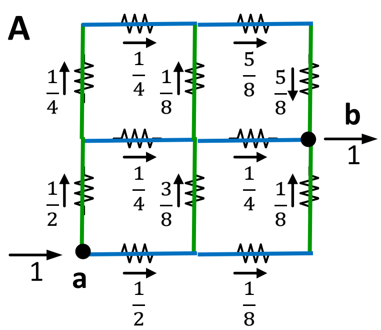
(iii) $j_{xy} = 0$ if x and y are not adjacent.

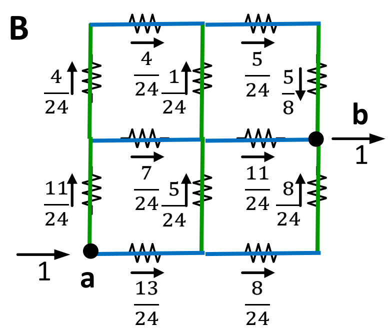



### b) Unit Current Flow and Unit Flow

If we apply a voltage between nodes a and b with $v_b = 0$ and set $v_a$ such that the current $i_a$ flowing into node a is 1, then the current $i_a$ flowing through the circuit that obeys fundamental electrical laws is called the <u>unit current flow</u> from a to b. Any other flow $i_{xy}$ from a to b for which $i_a = -i_b = 1$ is called a <u>unit flow</u>.

Diagram A below shows an example of a unit flow in a resistor network, while Diagram B shows the unit current flow for the same network as determined from circuit analysis.




### c) Thomson's Principle

A basic principle known as Thomson's Principle states that in an electrical network, unit current flow minimizes the energy dissipated over all other unit flows. This principle is stated more formally as

If i is the unit flow from a to b determined by Kirchoff's Laws, then the energy dissipation ½ $\sum_{Rxy} i_{xy}^2 R_{xy}$ minimizes the energy dissipation ½ $\sum_{Rxy} j_{xy}^2 R_{xy}$ among all unit flows j from a to b.

For example, if all the resistors have unit resistance, the energy dissipation for the unit flow in Diagram A above is

$$\tfrac{1}{2}\,[2(\tfrac{1}{2})^2 + 4(\tfrac{1}{4})^2 + 3(1/8)^2 + (3/8)^2 + 2(5/8)^2] = 55/64 = \mathbf{0.86},$$

while the energy dissipation for the unit current flow in Diagram B above is

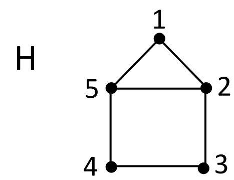
$$\tfrac{1}{2}\,[2(^{11}/_{24})^2 + (^{13}/_{24})^2 + (^{7}/_{24})^2 + 2(^{4}/_{24})^2 + (^{1}/_{24})^2 + 3(^{5}/_{24})^2 + 2(^{8}/_{24})^2] = 29/48 = \mathbf{0.60},$$

which is less than that of the unit flow, as expected. Refer to Doyle & Snell [[2], p. 63] for a proof of Thomson's Principle.

## III. Graph Theory Concepts

In this section we present some basic concepts in graph theory. For more detailed information, refer to "Algebraic Graph Theory" by Chris Godsil and Gordon Royle [**Error! Reference source not found.**].

A graph G consists of a vertex set V(G) and an edge set E(G), where an edge is an unordered pair of distinct vertices of G. In this paper we restrict our attention to simple graphs, those with no edges between a vertex and itself, with vertex set V(G) = {1, 2, …, n}. For example, for the following definitions, let H be the house graph with V(G) = {1, 2, 3, 4, 5} and E(G) = {(1, 2), (2, 3), (2, 5), (3, 4), (4, 5), (5, 1)}, as shown below.




### A. Types of Graphs

Some special types of graphs discussed in this paper are defined here.

**1. Path**
A path $P_n$ is a sequence of n distinct vertices $v_1, v_2, \ldots, v_n$ such that there is an edge between $v_i$ and $v_{i+1}$ for i = 1 to n-1.

For example, in the House graph above, the sequence 1, 2, 5, 4 is a path.

**2. Cycle**
A cycle $C_n$ is a path with $v_1 = v_n$.

For example, in the House graph above, the sequence 2, 3, 4, 5, 2 is a cycle.

**3. Complete Graph**
A complete graph $K_n$ is a graph with n vertices such that there is an edge between each pair of vertices.

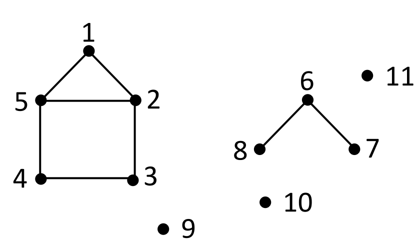
For example, in the House graph above, the triangle is a complete graph, while the square is not since it does not include the edges (2, 4) and (3, 5).

### B. Distance

The distance, $d_{xy}$, between two vertices x and y in a graph X is defined as the length of the shortest path from x to y. For example, in the House graph above, $d_{12} = 1$, $d_{13} = 2$, and $d_{24} = 2$.

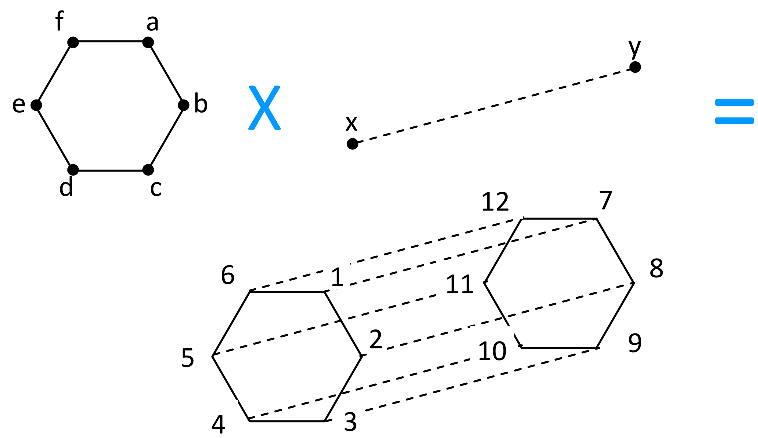
### C. Connected Graph

If there is a path between any two vertices of a graph X, then X is connected. For example, the House graph above is connected, while the 11-vertex graph shown below is not.



## D. Product Graph

The product graph, G x H, of graphs G and H has vertices V(G) x V(H) where two vertices (a, x) and (b, y) are adjacent, i.e. there is an edge between them, if either a = b and xy is an edge in H or x = y and ab is an edge in G. For example, the product of a cycle graph on 6 vertices and the complete graph with two vertices, $C_6$ x $K_2$, has 12 vertices and can be represented as follows:




Thus, we have the following vertices for the product graph G X H:

| | |
|---|---|
| 1 = (a, x) = ax | 7 = (a, y) = ay |
| 2 = (b, x) = bx | 8 = (b, y) = by |
| 3 = (c, x) = cx | 9 = (c, y) = cy |
| 4 = (d, x) = dx | 10 = (d, y) = dy |
| 5 = (e, x) = ex | 11 = (e, y) = ey |
| 6 = (f, x) = fx | 12 = (f, y) = fy |

## E. Orderable Graph

A graph H is orderable if there exists a sequence $x_1, x_2, \ldots, x_n$ of the vertices of H such that, for any sequence of real numbers $a_1 \le a_2 \le \cdots \le a_n$ and any permutation σ of {1, 2, …, n}, the assignment of $a_1, a_2, \ldots, a_n$ to the vertices with corresponding indices, $x_1, x_2, \ldots, x_n$, has the property

$$\sum_{x_ix_j \in E(H)} (a_i - a_j)^2 \le \sum_{x_ix_j \in E(H)} (a_{\sigma i} - a_{\sigma j})^2.$$

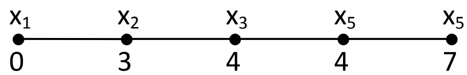
As we will prove in Lemma 2 and Lemma 3, paths and cycles are orderable graphs. A complete graph is also an orderable graph. All vertices are connected to every other vertex, so we will have equality:

$$\sum_{x_ix_j \in E(H)} (a_i - a_j)^2 = \sum_{x_ix_j \in E(H)} (a_{\sigma i} - a_{\sigma j})^2.$$

If a graph H is orderable, we refer to the sequence $x_1, x_2, \ldots, x_n$ as a voltage ordering for H.

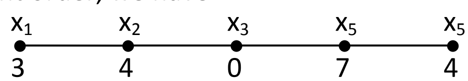
**Example:**

For the path $x_1, \ldots, x_5$ with the real numbers $a_1, a_2, \ldots, a_5$ = 0, 3, 4, 4, 7 assigned to its vertices in increasing order, we have




$$\sum_{x_ix_j \in E(H)} (a_i - a_j)^2 = 2(0-3)^2 + 2(3-4)^2 + 2(4-4)^2 + 2(4-7)^2 = 38,$$

and in a different order, we have




$$\sum_{x_ix_j \in E(H)} (a_{\sigma i} - a_{\sigma j})^2 = 2(3-4)^2 + 2(4-0)^2 + 2(0-7)^2 + 2(7-4)^2 = 150.$$

So, as expected for an orderable graph, we have

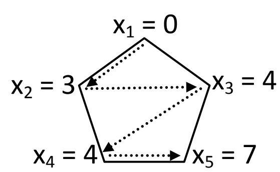
$$\sum_{x_ix_j \in E(H)} (a_i - a_j)^2 = 38 \le 150 = \sum_{x_ix_j \in E(H)} (a_{\sigma i} - a_{\sigma j})^2.$$

In fact, as we will see in the Proof of Theorem 1 using Thomson's Principle, 38 is the minimum value for any ordering of the vertices of this path.

**Example:**

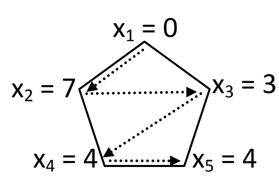
Consider the cycle on five vertices labeled with $x_1, \ldots, x_5$, each assigned one of the real numbers $a_1, a_2, \ldots, a_5$ = 0, 3, 4, 4, 7 as shown below.

For σ = identity permutation, $a_{\sigma i} = a_i$, we have




$$\sum_{x_ix_j \in E(H)} (a_{\sigma i} - a_{\sigma j})^2 = 2(4-0)^2 + 2(7-4)^2 + 2(4-7)^2 + 2(3-4)^2 + 2(3-0)^2 = 88.$$

For $\sigma = (2\ 5\ 4\ 3) = \binom{1\ 2\ 3\ 4\ 5}{1\ 5\ 2\ 3\ 4}$, we have




$$\sum_{x_ix_j \in E(H)} (a_{\sigma i} - a_{\sigma j})^2 = 2(3-0)^2 + 2(4-3)^2 + 2(4-4)^2 + 2(4-7)^2 + 2(7-0)^2 = 136.$$

In fact, as we will see in the Proof of Theorem 1 using Thomson's Principle, 88 is the minimum value for any ordering of the vertices of this cycle.

# IV. Theorems

We first present the main theorem, Theorem 1, which is a statement about the maximum effective resistance between two points in the cross product G x H, where G is any connected graph and H is a connected orderable graph. In order to better understand Theorem 1, in Corollary 1, Corollary 2, and Corollary 3 we examine examples of simpler cases where H is a single path, a single complete graph, or a single cycle.

The second theorem, Theorem 2, is a more general statement about the maximum effective resistance between two points in the cross product G x H, where G is any graph and H is a product of paths, complete graphs, and cycles.

The third theorem, Theorem 3, is a statement about the minimum effective resistance between two points in the cross product G x H, where G is any graph and H is a product of complete graphs. We follow this theorem with some "non-theorem" comments about the minimum effective resistance in other graph products.

## A. Theorem 1

Suppose we want to determine the maximum effective resistance between two points in a product graph. Using the notation R[(a,x),(b,y)] to denote the effective resistance between the points (a,x) and (b,y), we have the following theorem.

> Let H be a connected orderable graph, and let $x_1, \ldots, x_n$ be a voltage ordering of its vertices. Let G be any connected graph with distinct vertices a and b. Consider G x H. The resistance R[(a,$x_1$),(b,y)] is maximized over vertices y of H at y = $x_n$.

To better understand this theorem we consider examples of specific cases of Theorem 1 in Corollary 1, Corollary 2, and Corollary 3.

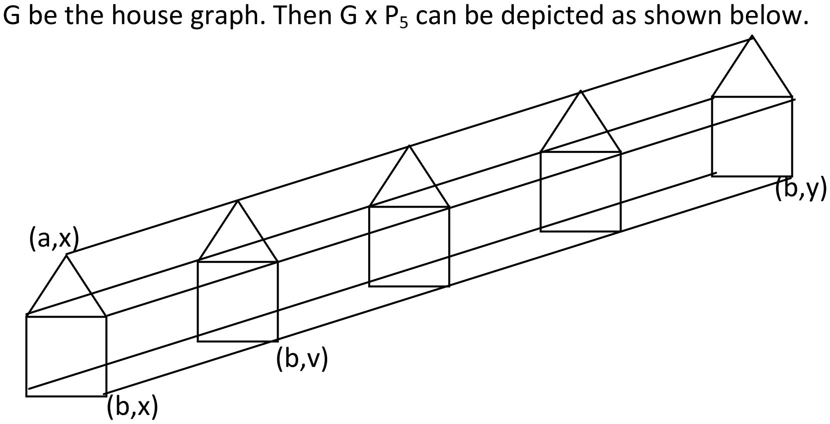
## B. Corollary 1

We will use the fact that paths are orderable to prove the following corollary:

> Let $P_n$ be an n-vertex path with endpoints x and y. Let a and b be any two distinct vertices of a graph G. Consider the graph G x $P_n$. The resistance R[(a,x),(b,v)] is maximized over vertices v of $P_n$ at v = y.

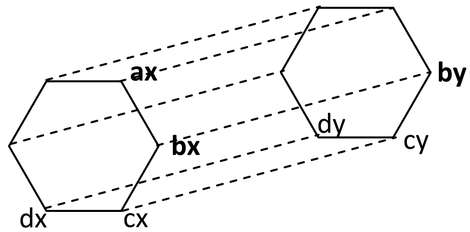
**Example:**

Let G be the house graph. Then G x $P_5$ can be depicted as shown below.



**Example:**

Let $G = C_6$ and $H = P_2$. Then, label $G \times H = C_6 \times P_2$ as shown below, simplifying vertices of the form (a, x) to ax.




| | | |
|---|---|---|
| R[(a, x),(b, x)] = R[1,2] = 0.64 | vs | R[(a, x),(b, y)] = R[1,8] = 0.78 |
| R[(a, x),(c, x)] = R[1,3] = 0.94 | vs | R[(a, x),(c, y)] = R[1,9] = 0.98 |
| R[(a, x),(d, x)] = R[1,4] = 1.04 | vs | R[(a, x),(d, y)] = R[1,10] = 1.06 |

These effective resistances can be determined using circuit analysis methods such as those described in Chapter II. Alternatively, they can be determined by methods using matrix representations of graphs. See Appendix 0.

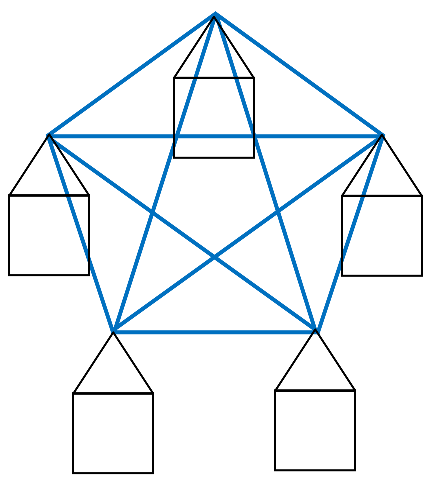
## C. <u>Corollary 2</u>

We will use the fact that complete graphs are orderable to prove the following corollary:

> Let $K_n$ be a complete graph with n vertices, and let x and y be any two distinct vertices of $K_n$. Let a and b be any two distinct vertices of a graph G. Consider the graph $G \times K_n$. Then R[(a,x),(b,**x**)] ≤ R[(a,x),(b,**y**)].

**Example:**

Let G be the house graph. Then $G \times K_5$ can be depicted as below, where, for clarity, only one of the five vertices on the house graph is shown connected to a $K_5$ graph.



**Example:**

Let $G = C_6$ and $H = K_2$. Then, since $K_2 = P_2$, the product graph $G \times H = C_6 \times K_2$ is the same as the second example for Corollary 1.

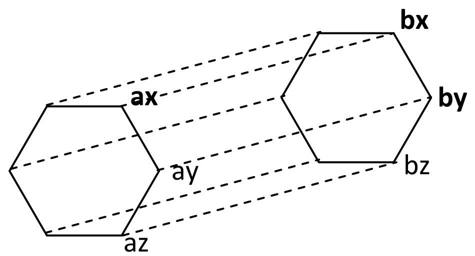
### D. Corollary 3

We will use the fact that complete graphs are orderable to prove the following corollary:

> Let $C_n$ be a cycle with n vertices, and let x, y, and z be three distinct vertices of $C_n$ with $d(x,y) \le d(x,z)$. Let a and b be any two distinct vertices of a graph G. Consider the graph G x $C_n$. Then $R[(a,x),(b,\mathbf{y})] \le R[(a,x),(b,\mathbf{z})]$.

**Example:**

Let G = $P_2$ and H = $C_6$. Then, label G x H = $P_2 \times C_6$ as shown, simplifying vertices of the form (a, x) to ax.




R[(a, x),(b, x)] = R[1,2] = 0.64 vs R[(a, x),(b, y)] = R[1,8] = 0.78
R[(a, x),(b, x)] = R[1,2] = 0.64 vs R[(a, x),(b, z)] = R[1,9] = 0.98
R[(a, y),(b, y)] = R[2,8] = 0.58 vs R[(a, y),(b, x)] = R[2,7] = 0.78

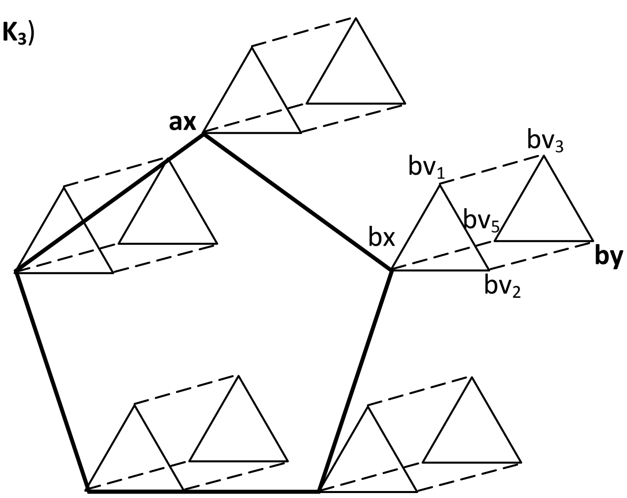
### E. Theorem 2

A theorem for determining the maximum effective resistance between two points in a more general product graph is the following.

> Let H be an arbitrary product of paths, complete graphs, and cycles. Let x and y be two vertices at maximum distance in H. Let a and b be distinct vertices of a graph G, and consider G x H. Then R[(a,x),(b,v)] is maximized over vertices v of H at v = y.

**Example:**

G x H = $C_5 \times (\mathbf{P_2} \times \mathbf{K_3})$



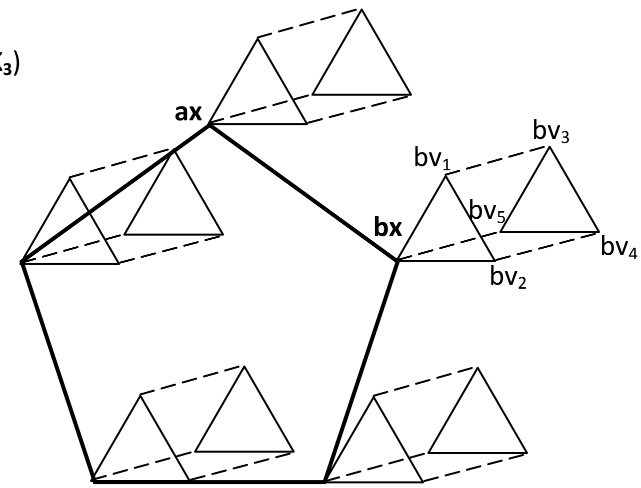
## F. Theorem 3

Suppose we want to determine the minimum effective resistance between two points in a product graph. Then, we have the following theorem for products with complete graphs.

> Let H be an arbitrary product of complete graphs, say $K_n$ x ··· x $K_p$, and let x be a vertex in H. Let a and b be distinct vertices of a graph G, and consider G x H. Then R[(a,x),(b,v)] is minimized over vertices v of H at v = x.

**Example:**

G x H = $C_5$ x (**$K_2$** x **$K_3$**)

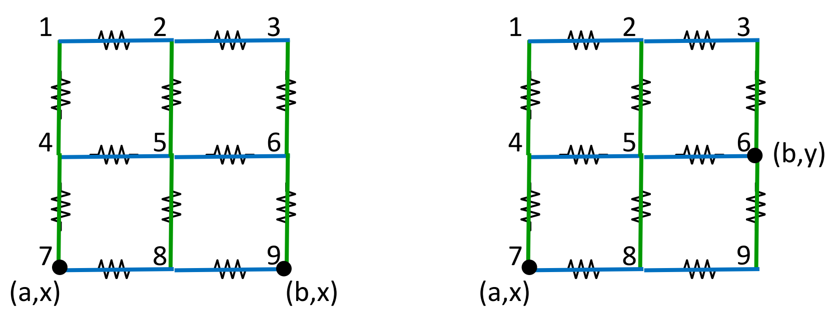



## G. NON-Theorem

To illustrate that the claims of the three theorems are not as obvious as they may seem, we provide an example to show Theorem 3 does not always hold if H is a path instead of a product of complete graphs.

If G x H = $P_3$ x $P_3$ with endpoint x in H, we can show that for some (a,x) and (b,v), R(a,x),(b,v)] is not minimized over vertices v of P at v = x.

Consider the product **G** x **H** = **$P_3$** x **$P_3$** with vertices labeled as shown below and with unit resistors between them.




Since (b, x) appears to be closer to (a, x) than (b, y) is to (a, x), we might expect that R[(a,x),(b,x)] ≤ R[(a,x),(b,y)]. But, from the circuit analysis example in Chapter II, R[(a,x),(b,**x**)] = 5/4 = 30/24. Following a similar process, as shown in Appendix 0, we can determine that R[(a,x),(b,**y**)] = 29/24. So, we have an example where R[(a,x),(b,v)] is not the minimum at v = x.

# V. Proofs of Theorems

**Lemma 1** is used to prove **Theorem 1** and **Lemma 3**. **Lemma 2** is used to show that **Corollary 1 i**s a special case of Theorem 1. **Lemma 3** is used to show that **Corollary 3** is another special case of Theorem 1. **Corollary 1**, **Corollary 2**, and **Corollary 3** and **Theorem 3** are used to prove **Theorem 2** and **Theorem 3**.

See Appendix 0 for a flowchart of the dependencies of the results.

## A. Lemma 1

The following lemma is used in the Proof of Theorem 1 in analyzing the minimum energy of a product graph given a particular assignment of voltages to vertices.

Given any two sequences $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ of real numbers, and any permutation σ of {1, 2, …, n},

$$\sum_{i=1}^{n} (a_i - b_i)^2 \le \sum_{i=1}^{n} (a_i - b_{\sigma i})^2.$$

**Example:**

Let the sequences $a_1, a_2, a_3, a_4, a_5$ = 1, 2, 2, 5, 10 and $b_1, b_2, b_3, b_4, b_5$ = 3, 4, 5, 5, 8, and let σ = (1 3)(4 5). Then, we have

$$\sum_{i=1}^{n} (a_i - b_i)^2 = (1-3)^2 + (2-4)^2 + (2-5)^2 + (5-5)^2 + (10-8)^2 = 23, \text{ and}$$

$$\sum_{i=1}^{n} (a_i - b_{\sigma i})^2 = (1-5)^2 + (2-4)^2 + (2-3)^2 + (5-8)^2 + (10-5)^2 = 55.$$

As expected, the sum when σ is the identity is less than that when σ is not the identity.

Proof

Let $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ be two sequences of real numbers and let σ be a permutation of {1, 2, …, n}. Suppose $a_i \le a_j$ and $b_k \le b_l$ for some i, j, k, l ∈ {1, 2, …, n}. Then

$$(a_j - a_i)(b_l - b_k) \ge 0,$$

which implies

$$a_i b_k + a_j b_l \ge a_i b_l + a_j b_k \tag{1}$$

Then,

$$\begin{aligned}(a_i - b_l)^2 + (a_j - b_k)^2 &= a_i^2 + b_k^2 + a_j^2 + b_l^2 - 2(a_i b_l + a_j b_k)\\ &\ge a_i^2 + b_k^2 + a_j^2 + b_l^2 - 2(a_i b_k + a_j b_l) && \text{by equation (1)}\\ &= (a_i - b_k)^2 + (a_j - b_l)^2. && (2)\end{aligned}$$

If $b_{\sigma 1} \ne b_1$, then there exists m ∈ {1, 2, …, n} such that $b_{\sigma m} = b_1$. So, $b_{\sigma m} < b_{\sigma 1}$. Then, by equation (2),

$$(a_1 - b_{\sigma 1})^2 + (a_m - b_{\sigma m})^2 \ge (a_1 - b_{\sigma m})^2 + (a_m - b_{\sigma 1})^2. \tag{3}$$

So,

$$\begin{aligned}\sum_{i=1}^{n} (a_i - b_{\sigma i})^2 &= (a_1 - b_{\sigma 1})^2 + (a_2 - b_{\sigma 2})^2 + \cdots + (a_m - b_{\sigma m})^2 + \cdots + (a_n - b_{\sigma n})^2\\ &\ge (a_1 - b_{\boldsymbol{\sigma m}})^2 + (a_2 - b_{\sigma 2})^2 + \cdots + (a_m - b_{\boldsymbol{\sigma 1}})^2 + \cdots + (a_n - b_{\sigma n})^2 && \text{by equation (3)}\\ &= (a_1 - b_{\mathbf{1}})^2 + (a_2 - b_{\sigma 2})^2 + \cdots + (a_m - b_{\sigma 1})^2 + \cdots + (a_n - b_{\sigma n})^2.\end{aligned}$$

Similarly, if $b_{\sigma 2} \neq b_2$, then there exists $p \in \{1, 2, \ldots, m-1, m+1 \ldots, n\}$ such that $b_{\sigma p} = b_2$. So, $b_{\sigma p} < b_{\sigma 2}$. Then,

$$\sum_{i=1}^{n} (a_i - b_{\sigma i})^2 = (a_1 - b_1)^2 + (a_2 - b_{\sigma 2})^2 + \cdots + (a_p - b_{\sigma p})^2 + \cdots + (a_n - b_{\sigma n})^2$$

$$\geq (a_1 - b_1)^2 + (a_2 - b_{\boldsymbol{\sigma p}})^2 + \cdots + (a_p - b_{\boldsymbol{\sigma 2}})^2 + \cdots + (a_n - b_{\sigma n})^2 \qquad \text{by equation (2)}$$

$$= (a_1 - b_1)^2 + (a_2 - b_{\mathbf{2}})^2 + \cdots + (a_m - b_{\sigma 1})^2 + \cdots + (a_n - b_{\sigma n})^2.$$

Continuing the process for each $b_{\sigma i}$, we end with

$$\sum_{i=1}^{n} (a_i - b_{\sigma i})^2 \geq (a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2$$

$$= \sum_{i=1}^{n} (a_i - b_i)^2. \qquad \square$$

## B. Proof of Theorem 1

Recall Theorem 1:

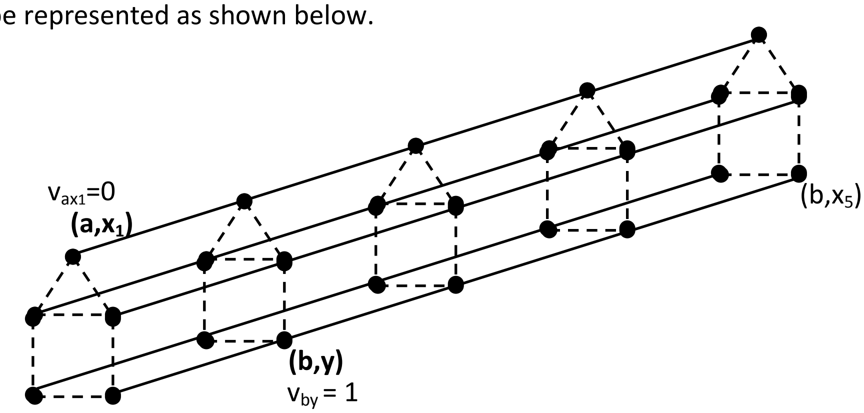
> Let **H** be a **connected orderable** graph, and let $\mathbf{x_1, \ldots, x_n}$ be a **voltage ordering** of its vertices. Let **G** be any **connected** graph with distinct vertices a and b. Consider G x H. The resistance $R[(a,x_1),(b,y)]$ is maximized over vertices v of H at $y = x_n$.

Let y be any vertex of H, and consider the voltages in G x H associated with a flow of electric current i from $(a, x_1)$ at voltage 0 to $(b, y)$ at voltage 1. The energy supplied is the reciprocal of the effective resistance between $(a, x_1)$ and $(b, y)$.

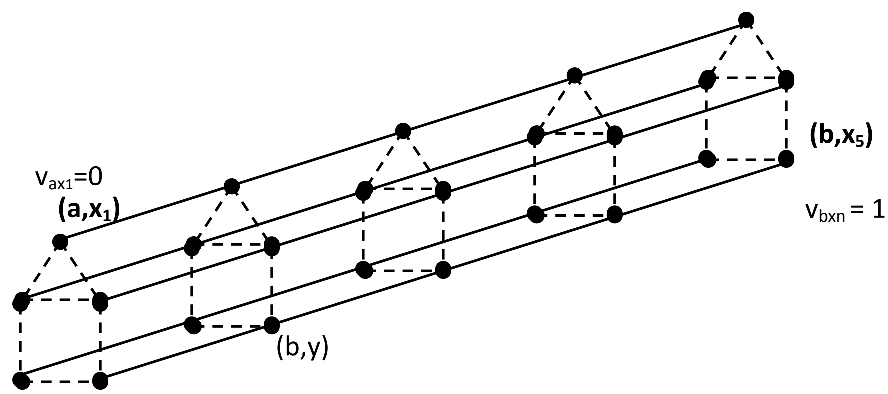
$$E_s = (v_{by} - v_{ax1})i = (v_{by} - v_{ax1})^2/R_{eff} = (1 - 0)^2/R_{eff} = 1/R_{eff}$$

So, $R[(a,x_1),(b,y)] = R_{eff} = 1/E_s = 1/E_d$, by the conservation of energy. By Thomson's Principle, $E_d$ is the minimum energy dissipated, so the effective resistance, $R[(a,x_1),(b,y)]$, is the maximum.

To illustrate these ideas, we will let G = House Graph and H = $P_5$. Then, the product House x $P_5$ can be represented as shown below.

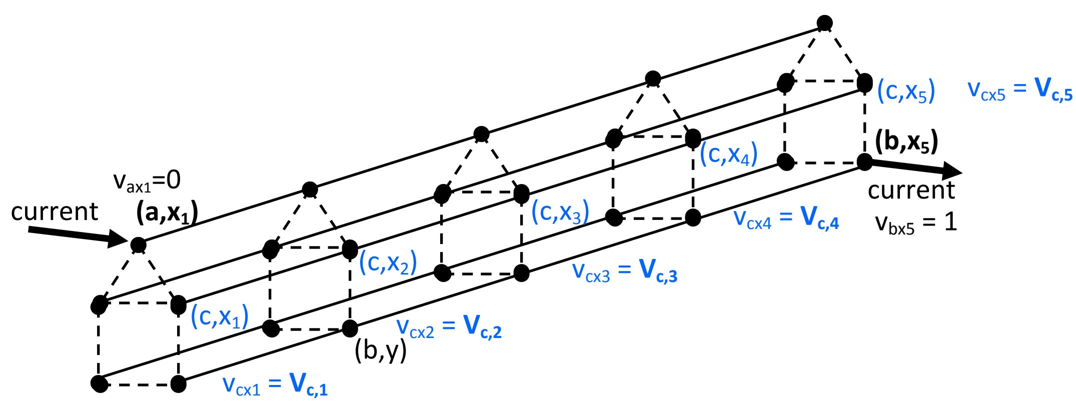


If we can construct a system of voltages of no greater energy with $(a, x_1)$ at voltage 0 and $(b, x_n)$ at voltage 1, we will have shown that $R[(a,x_1),(b,x_n)]$ = 1/Energy is at maximized at $y = x_n$. Note that we have used the term "network" to refer to an assignment of voltages to vertices that obeys the laws of physics (Ohm's Law, Kirchoff's Laws, and conservation of energy). We now introduce the term "system" to refer to an arbitrary assignment of voltages to vertices.



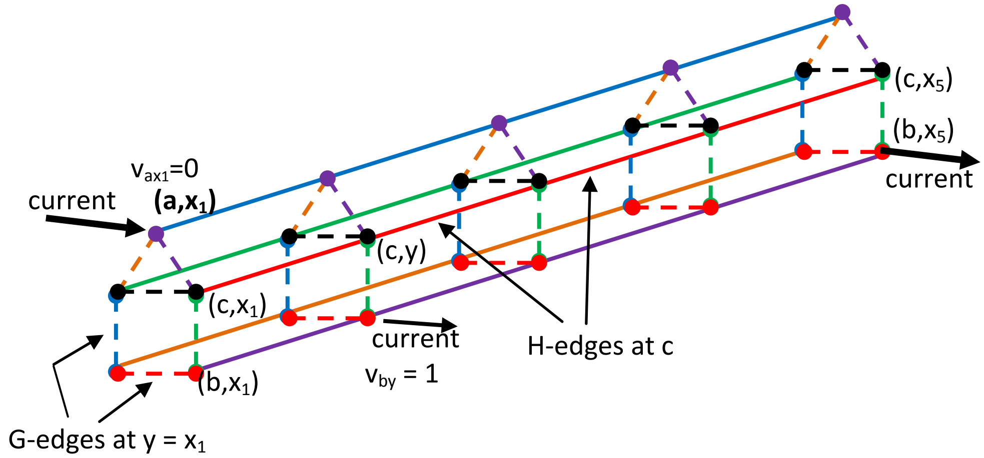

Because all current enters the network at vertex $(a, x_1)$ and leaves at vertex $(b, x_n)$, which are assigned voltage of 0 and 1, respectively, the voltages at all other vertices are between 0 and 1. For each vertex c of G, consider the n voltages $V_{c,1} \leq V_{c,2} \leq \ldots \leq V_{c,n}$ associated with vertices (c,y) of the product where y is an integer between 1 and n. We construct our new system by rearranging these voltages so that vertex $(c, x_i)$ has voltage $V_{c,i}$ for each $c \in V(G)$ and i = 1, …, n, that is, by assigning the voltages in ascending order in correspondence with the voltage ordering of the orderable graph H. We claim that this new system has no greater energy than before.

For our House x $P_5$ example, we arrange the voltages $V_{c,1}$ , …, $V_{c,n}$ corresponding to vertices $(c, x_1)$, …, $(c, x_5)$ as shown below.

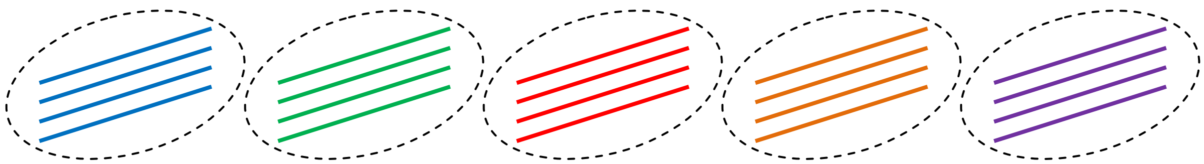



We partition the edge set of G x H as follows. First, we introduce the term "H-edges at c" to mean edges of G x H of the form (c, x)(c, y) with $xy \in E(H)$ and the term "G-edges at y" to mean edges of G x H of the form (c, y)(d, y) with $cd \in E(G)$. For each vertex c of G, we consider all the H-edges at c together. For each vertex y of H, we consider all the G-edges at y together. This accounts for all the edges of the product.

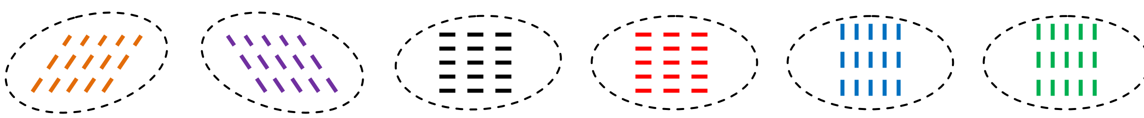
For our House x $P_5$ example, there are 5 partitions of H-edges at c and 6 partitions of G-edges for a total of 11 partitions as shown below.




H-edge Partitions:



G-edge Partitions:



For each vertex c of G (the House graph in our example), the energy arising from the H-edges at c (edges in the path $P_5$) has not increased since we have rearranged the voltages according to the given voltage ordering, and by the fact that $P_5$ is an orderable graph, we have achieved the minimum possible energy from this set of voltages. In our example, we have

$$\sum_{x_i x_j \in E(H)} (V_{c,i} - V_{c,j})^2 \le \sum_{x_i x_j \in E(H)} (V_{c,\sigma i} - V_{c,\sigma j})^2.$$

For each vertex y of H (the path), the energy arising from the G-edges at y (edges in the house graph) is also no greater than before, by Lemma 1, since there is a correspondence between the order of the voltages of the vertices (c, y) and the order of the voltages of vertices (d, y) for y = $x_1, x_2, \ldots, x_n$. In our example, the sequences $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ are $V_{c,1} \le V_{c,2} \le \cdots \le V_{c,n}$ and $V_{b,1} \le V_{b,2} \le \cdots \le V_{b,n}$. Thus, we have

$$\sum_{i=1}^{n} (V_{c,i} - V_{b,i})^2 \le \sum_{i=1}^{n} (V_{c,i} - V_{b,\sigma i})^2.$$

Thus, the total energy of the system is not increased, as required. So, the effective resistance is not decreased. Therefore, the effective resistance is at a maximum at y = $x_n$.

□

## C. Lemma 2

Let $P_n$ be the path $x_1, x_2, \ldots, x_n$ on n vertices. The order $x_1, x_2, \ldots, x_n$ is a voltage ordering of the vertices of $P_n$.

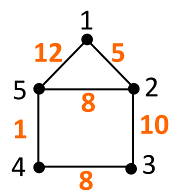
Proof

We need to show that, for any sequence of real numbers $a_1 \le a_2 \le \cdots \le a_n$ and any permutation σ of {1, 2, …, n}, the assignment of $a_1, a_2, \ldots, a_n$ to the vertices with corresponding indices, $x_1, x_2, \ldots, x_n$, has the property

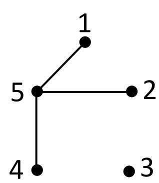
$$\sum_{x_i x_j \in E(P_n)} (a_i - a_j)^2 \le \sum_{x_i x_j \in E(P_n)} (a_{\sigma i} - a_{\sigma j})^2.$$

This requires the use of Prim's Algorithm, but before describing the algorithm, we need to define some more terms from graph theory.

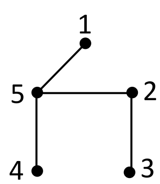
A weighted graph is a graph with a number assigned to each edge.



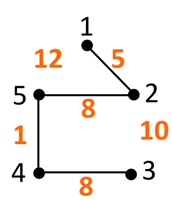

A tree is a connected graph with no cycles.




A spanning tree of a graph is a tree that touches all the vertices (so, it only makes sense in a connected graph).




A minimum spanning tree is a spanning tree whose sum of edge weights is as small as possible.




Prim's Algorithm grows a spanning tree from a single vertex of a connected weighted graph, iteratively adding the edge with least weight from a vertex already reached to a vertex not yet reached, finishing when all the vertices of X have been reached. (Ties are broken arbitrarily.) Prim's Algorithm produces a minimum-weight spanning tree. For an induction proof of this statement, see "Applied Combinatorics" by Alan Tucker [[5]].

The steps for finding a voltage ordering of $P_n$ using Prim's Algorithm are as follows:

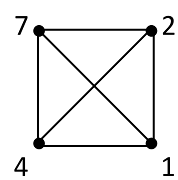
1. Label the vertices of the complete graph $K_n$ with the numbers $a_1, a_2, \ldots, a_n$.
2. Label each edge $a_ia_j$ with the weight $(a_i - a_j)^2$.
3. Color the edge with the smallest weight that is connected to the vertex labeled $a_1$. This edge is in the minimum spanning tree.
4. Color the edge with the smallest weight that is connected to the tree, but has one vertex not in the tree.
5. Repeat Step 4 until all vertices in $K_n$ are included in the tree.

The minimal spanning tree that Prim's Algorithm produces will be a path with the sequence $x_1, x_2, \ldots, x_n$ of vertices corresponding to the numbers $a_1, a_2, \ldots, a_n$ a voltage ordering of that path.

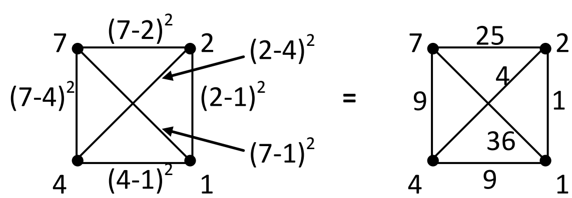
□

As an example, we find a voltage ordering of $P_4$ for the sequence 1, 2, 4, 7.

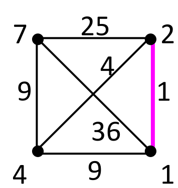
1. Label the vertices of the complete graph $K_4$ with the numbers 1, 2, 4, 7.



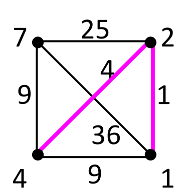

2. Label each edge with its weight.



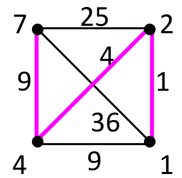

3. Color the edge with weight 1 since it is the edge with smallest weight that is connected to the vertex labeled 1. This edge is in the minimum spanning tree.




4. Color the edge with weight 4 since it is the edge with the smallest weight that is connected to the tree, but not in the tree.



5. Color one of the edges with weight 9 since they are the edges with the smallest weight that is connected to a vertex in the tree, but is not in the tree.




This minimal spanning tree is a path, so labeling the vertices of path $x_1$, $x_2$, $x_3$, $x_4$ in ascending order of the assigned numbers 1,2 4, 7, gives us $x_1$, $x_2$, $x_3$, $x_4$ as a voltage ordering of $P_4$.

## D. Proof of Corollary 1

Since Corollary 1 is a special case of Theorem 1 where the connected orderable graph H is a path $P_n$, its proof follows from the proof of Theorem 1. □

## E. Proof of Corollary 2

Since Corollary 2 is a special case of Theorem 1 where the connected orderable graph H is a complete graph $K_n$, its proof follows from the proof of Theorem 1. □

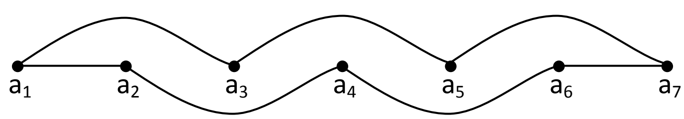
## F. Lemma 3

Let $C_n$ be the cycle $x_1, x_2, \ldots, x_n, x_1$ on n vertices. The order $x_1, x_n, x_2, x_{n-1}, x_3, x_{n-2}, \ldots, x_{[n+1/2]}$ is a voltage ordering of the vertices of $C_n$.

Proof

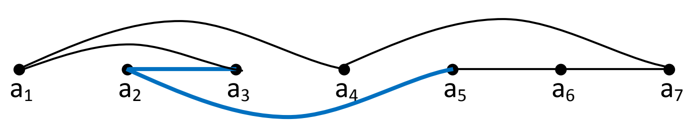
Let $a_1 \leq a_2 \leq \ldots \leq a_n$ be numbers, considered as points on the real line. We claim that a cycle through these points minimizing the sum of the squares of the edge-lengths, $\sum_{i=1}^{n} (a_{i+1} - a_i)^2$, is the one with the edges $a_1a_2$, $a_{n-1}a_n$, and $a_ia_{i+2}$ for i = 1, ..., n – 2. For example, for n = 7, we have

(a)

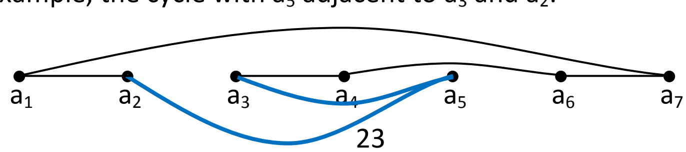



If the $a_i$ are not distinct, and since $a_1 \leq a_2 \leq \ldots \leq a_n$, then $a_j = a_{j+1}$ for some j. Thus, $(a_{j+1} - a_j)^2 = 0$ and therefore does not add to the sum of the squares of the edge-lengths. So, we will assume the $a_i$ are distinct. Any other cycle has one of the following features:

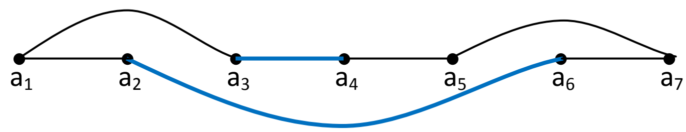
(b) for some i > 1, $a_i$ is adjacent to two vertices $a_j$ and $a_k$, with i < j, k;
For example, the cycle with $a_2$ adjacent to $a_3$ and $a_5$:



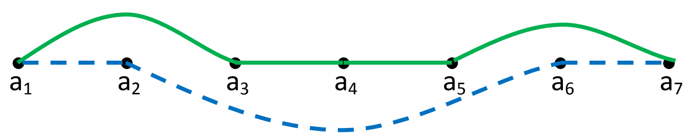

(c) for some i < n, $a_i$ is adjacent to two vertices $a_j$ and $a_k$, with i > j, k;
For example, the cycle with $a_5$ adjacent to $a_3$ and $a_2$:



(d) for some $i < j < k < l$, $a_i a_l$ and $a_j a_k$ are both edges of the cycle;
For example, the cycle with $a_i a_l = a_2 a_6$ and $a_j a_k = a_3 a_4$:



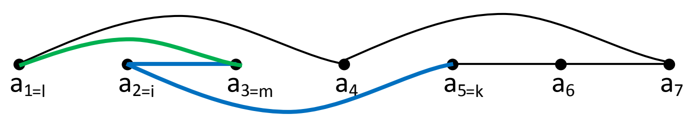

If neither (b) nor (c) holds, then, for every $1 < i < n$, $a_i$ must be adjacent to a vertex $a_j$ where $i > j$ and a vertex $a_k$ where $i < k$. Then, the cycle can be decomposed into two monotone paths from $a_1$ to $a_n$, and if either path "jumps" over more than one vertex, then all the vertices inside the jump are picked up by the other path, giving case (d).

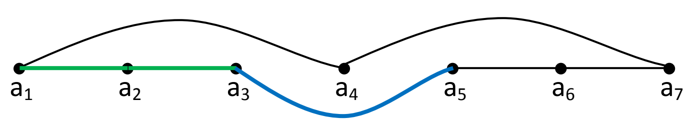



It remains to be seen that (b), (c), and (d) all give non-optimal cycles.

In case (b), there must be some edge of the cycle $a_l a_m$ with $l < i < m$. We now change the cycle by replacing edge $a_l a_m$ by the path $a_l a_i a_m$, and replacing the path $a_j a_i a_k$ by the edge $a_j a_k$.

In the example for (b) given above, we can replace $a_1 a_3$ with $a_1 a_2 a_3$ and $a_3 a_2 a_5$ with $a_3 a_5$:




Both changes decrease the sum of the squares of the edge lengths as we prove below.

Claim

For for $a \le b \le c$,
(i) $(c - b)^2 + (b - a)^2 \le (c - a)^2$, and
(ii) $(c - b)^2 \le (c - a)^2 + (b - a)^2$.

Proof

Let $a \le b \le c$. Then,

$$\begin{aligned} b &\ge a \\ b\mathbf{(c-b)} &\ge a\mathbf{(c-b)} && \text{since } c \ge b \\ bc - b^2 &\ge ac - ab \\ bc - b^2 \mathbf{+ ab} &\ge ac \\ \mathbf{-2}bc \mathbf{+ 2}b^2 \mathbf{- 2}ab &\mathbf{\le -2}ac \\ \mathbf{c^2} - 2bc + \mathbf{b^2 + b^2} - 2ab \mathbf{+ a^2} &\mathbf{\le c^2} - 2ac \mathbf{+ a^2} \\ (c-b)^2 + (b-a)^2 &\le (c-a)^2. && \text{(A)} \end{aligned}$$

Further decreasing the left hand side by subtracting $(b - a)^2$ and further increasing the right hand side by adding $(b - a)^2$, we get

$$(c - b)^2 \leq (c - a)^2 + (b - a)^2. \quad \text{(B)}$$

Equation (A) satisfies (i) and equation (B) satisfies (ii). □

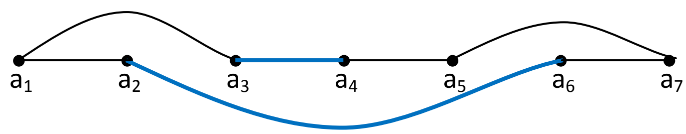
If we let $a = a_l$, $b = a_i$, and $c = a_m$, equation (A) becomes $(a_m - a_i)^2 + (a_i - a_l)^2 \leq (a_m - a_l)^2$. Thus, in replacing edge $a_l a_m$ by the path $a_l a_i a_m$, we decrease the sum of the squares of the edge lengths.

If we let $a = a_i$, $b = a_m$, and $c = a_k$, equation (B) becomes $(a_k - a_m)^2 \leq (a_k - a_i)^2 + (a_m - a_i)^2$. Thus, in replacing the path $a_j a_i a_k$ by the edge $a_j a_k$, we also decrease the sum of the squares of the edge lengths.

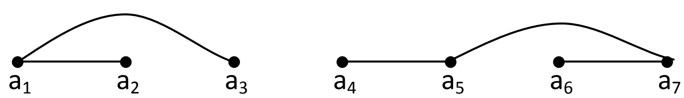
Case (c) is symmetric. Thus, case (b) and case (c) are not optimal because reconnecting them to form case (d) decreases the sum of the squares of the edge-lengths, by the claim above.

In case (d), deleting the edges $a_i a_l$ and $a_j a_k$ from the cycle forms two paths. These can be reconnected to form a cycle either by adding edges $a_i a_j$ and $a_k a_l$ or by adding edges $a_i a_k$ and $a_j a_l$.

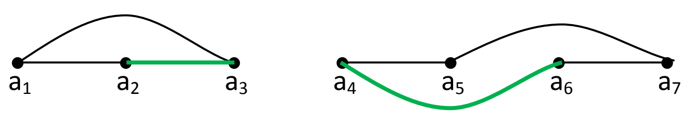
For the example given above for case (d),



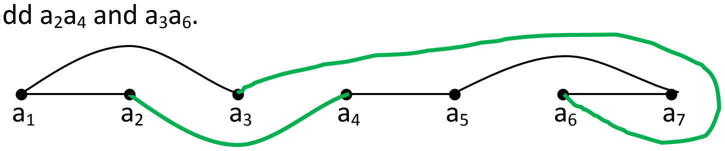

we delete $a_2 a_6$ and $a_3 a_4$,



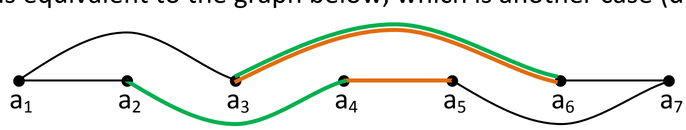

and, since adding $a_2 a_3$ and $a_4 a_6$ does not form a cycle,



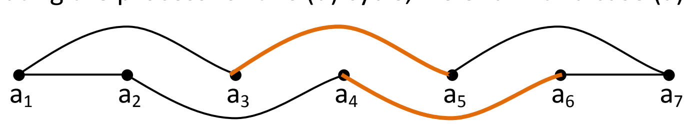

we add $a_2 a_4$ and $a_3 a_6$.




This is equivalent to the graph below, which is another case (d) cycle.




Repeating the process for this (d) cycle, we end with a case (a) cycle.




Thus, case (d) is not optimal because reconnecting it to form case (a) decreases the sum of the squares of the edge-lengths, by Lemma 1. To illustrate the application of Lemma 1, consider the following example.

Suppose we have the two sequences $c_1 \le c_2$ and $d_1 \le d_2$, where

$c_1 = a_3 = 3,$
$c_2 = a_6 = 6,$
$d_1 = a_4 = 4,$ and
$d_2 = a_5 = 5.$

Then,

$(c_1 - \mathbf{d_1})^2 + (c_2 - \mathbf{d_2})^2 = (3 - 4)^2 + (6 - 5)^2 = 2$, which is less than
$(c_1 - \mathbf{d_2})^2 + (c_2 - \mathbf{d_1})^2 = (3 - 5)^2 + (6 - 4)^2 = 8.$

Therefore, we have shown that the only cycle through the vertices of $C_n$ that minimizes the sum of the squares of the edge-lengths is case (a) with the voltage ordering

$$x_1, x_n, x_2, x_{n-1}, x_3, x_{n-2}, \ldots, x_{\lceil n+1/2 \rceil}.$$

□

## G. Proof of Corollary 3

Since Corollary 3 is a special case of Theorem 1 where the connected orderable graph H is a cycle $C_n$, which is orderable by Lemma 3, its proof follows from the proof of Theorem 1. □

## H. Proof of Theorem 2

Recall Theorem 2:

> Let H be an arbitrary product of paths, complete graphs, and cycles. Let x and y be two vertices at maximum distance in H. Let a and b be distinct vertices of a graph G, and consider G x H. Then R[(a,x),(b,v)] is maximized over vertices v of H at v = y.

Let $H = H_1 \times H_2 \times \cdots \times H_k$ be a product of paths, cycles, and complete graphs. Let $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ be vertices at maximum distance in H. Note that $x_i$ and $y_i$ are at maximum distance in $H_i$ for all i. Let $v = (v_1, \ldots, v_k)$ be any vertex of H. For j = 0, …, k, let $v^j$ be the vertex $(y_1, \ldots, y_j, v_{j+1}, \ldots, v_k)$, so that $v^0 = v$ and $v^k = y$.

$v^0 = (v_1, v_2, v_3, \ldots, v_{k-2}, v_{k-1}, v_k) = v$

$v^1 = (\mathbf{y}_1, v_2, v_3, \ldots, v_{k-2}, v_{k-1}, v_k)$

$v^2 = (\mathbf{y}_1, \mathbf{y}_2, v_3, \ldots, v_{k-2}, v_{k-1}, v_k)$

$\ddots$

$v^{k-2} = (\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots, \mathbf{y}_{k-2}, v_{k-1}, v_k)$

$v^{k-1} = (\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots, \mathbf{y}_{k-2}, \mathbf{y}_{k-1}, v_k)$

$v^k = (\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots, \mathbf{y}_{k-2}, \mathbf{y}_{k-1}, \mathbf{y}_k) = y$

We claim that, for each j, $R[(a,x),(b,v^{j-1})] \le R[(a,x),(b,v^j)]$. This implies that $R[(a,x),(b,v)] \le R[(a,x),(b,y)]$, which is the desired result.

Note that the only co-ordinate in which the vertices $(b, v^{j-1})$ and $(b, v^j)$ differ is that corresponding to $H_j$. Thus, we may regard the graph G x H as the product $(G \times H_1 \times \cdots \times H_{j-1}, H_{j+1} \times \cdots \times H_k) \times H_j$, and apply **Corollary 1**, **Corollary 2**, or **Corollary 3**, as appropriate, to establish the claim.

□

## I. Proof of Theorem 3

Recall Theorem 3:

> Let H be an arbitrary product of complete graphs, say $K_n$ x ⋯ x $K_p$, and let x be a vertex in H. Let a and b be distinct vertices of a graph G, and consider G x H. Then R[(a,x),(b,v)] is <u>minimized</u> over vertices v of H at v = x."

The proof is similar to that given for Theorem 2.

Let H = $K_1$ x $K_2$ x ⋯ x $K_k$ be a product of complete graphs. Let x = ($x_1$, …, $x_k$) and y = ($y_1$, …, $y_k$) be vertices at maximum distance in H. Note that $x_i$ and $y_i$ are at maximum distance in $H_i$ for all i. Let v = ($v_1$, …, $v_k$) be any vertex of H. For j = 0, …, k, let $v^j$ be the vertex ($y_1$, …, $y_j$, $v_{j+1}$, …, $v_k$), so that $v^0$ = v and $v^k$ = y, as in the proof of Theorem 2.

We claim that, for each j, R[(a,x),(b,$v^{j-1}$)] ≤ R[(a,x),(b,$v^j$)]. This implies that R[(a,x),(b,v)] ≥ R[(a,x),(b,x)], which is the desired result.

Note that the only co-ordinate in which the vertices (b, $v^{j-1}$) and (b, $v^j$) differ is that corresponding to $H_j$. Thus, we may regard the graph G x H as the product (G x $K_1$ x ⋯ x $K_{j-1}$, $K_{j+1}$ x ⋯ x $K_k$) x $K_j$, and apply Corollary 2 to establish the claim.

□

## VI. Summary

In this paper we used graph theory concepts to learn some of the behaviors of voltages and currents in several resistor networks. This understanding of the behavior in elementary circuits can lead to insights into the behavior in more complex circuits.

We proved a statement about the maximum effective resistance between two points in the cross product G x H, where G is any connected graph and H is a connected orderable graph. We also proved a more general statement about the maximum effective resistance between two points in the cross product G x H, where G is any graph and H is a product of paths, complete graphs, and cycles. We showed that although we could make a statement about the minimum effective resistance between two points in the cross product G x H, where G is any graph and H is a product of complete graphs, the statement does not always hold if H is not a product of complete graphs.

We used circuit analysis methods to determine voltages and currents that exhibit minimum energy, and thus maximum effective resistances, between two nodes. An alternative method that avoids complicated circuit analysis uses various matrices to represent resistor networks that can be interpreted as product graphs. A description of this method and examples are provided in the appendix.

Some other questions to consider are: What graphs, in addition to paths, cycles, and complete graphs, are orderable? Can we make similar statements about any unorderable graphs? Are these or similar results applicable to electrical properties in addition to effective resistance and energy? What else can these results tell us about the flow of electric current in a circuit? These and many other questions are left to the reader for further exploration.

# References


[1] B. Bollobás and G. Brightwell. Random Walks and Electrical Resistances in Products of Graphs. *Discrete Applied Mathematics*, 73: 69-79, 1997.

[2] J. W. Nilsson and S. A. Riedel. *Electric Circuits*. 7th Edition, Prentice Hall, New Jersey, 2005.

[3] P. G. Doyle and J. L. Snell. *Random Walks and Electric Networks*. MAA, Washington, DC, 1984.

[4] C. D. Godsil and G. Royle. *Algebraic Graph Theory*. Springer, New York, 2001.

[5] A. Tucker. *Applied Combinatorics*. Wiley, Hoboken, NJ, 1984.

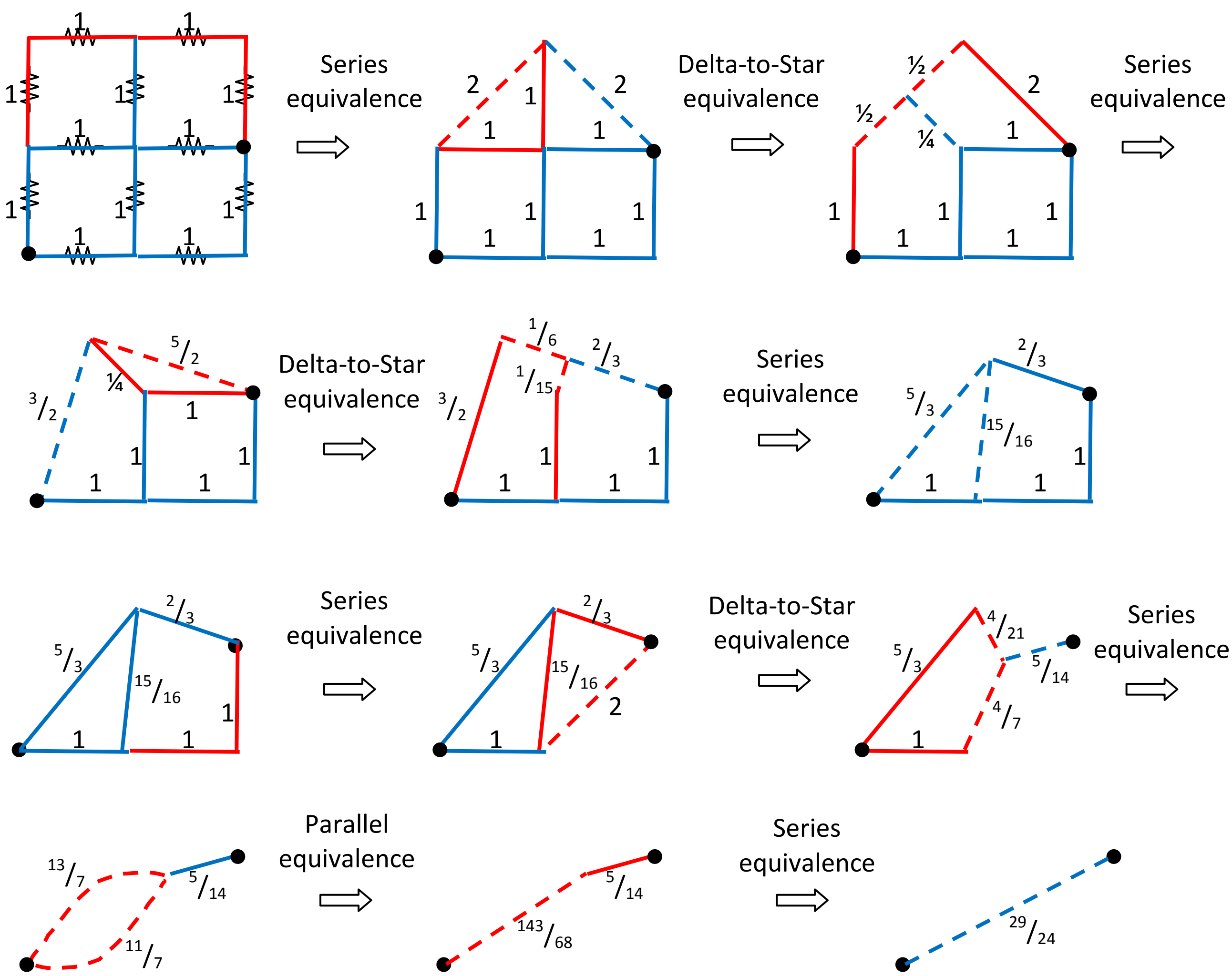
[6] D. A. Spielman, *Lectures 12 and 13*. Web page URL http://www.cs.yale.edu/homes/spielman/462/, accessed July 2012.

[7] D. A. Spielman, *Lectures 8, 9, and 10*. Web page URL http://www.cs.yale.edu/homes/spielman/462/, accessed July 2012.

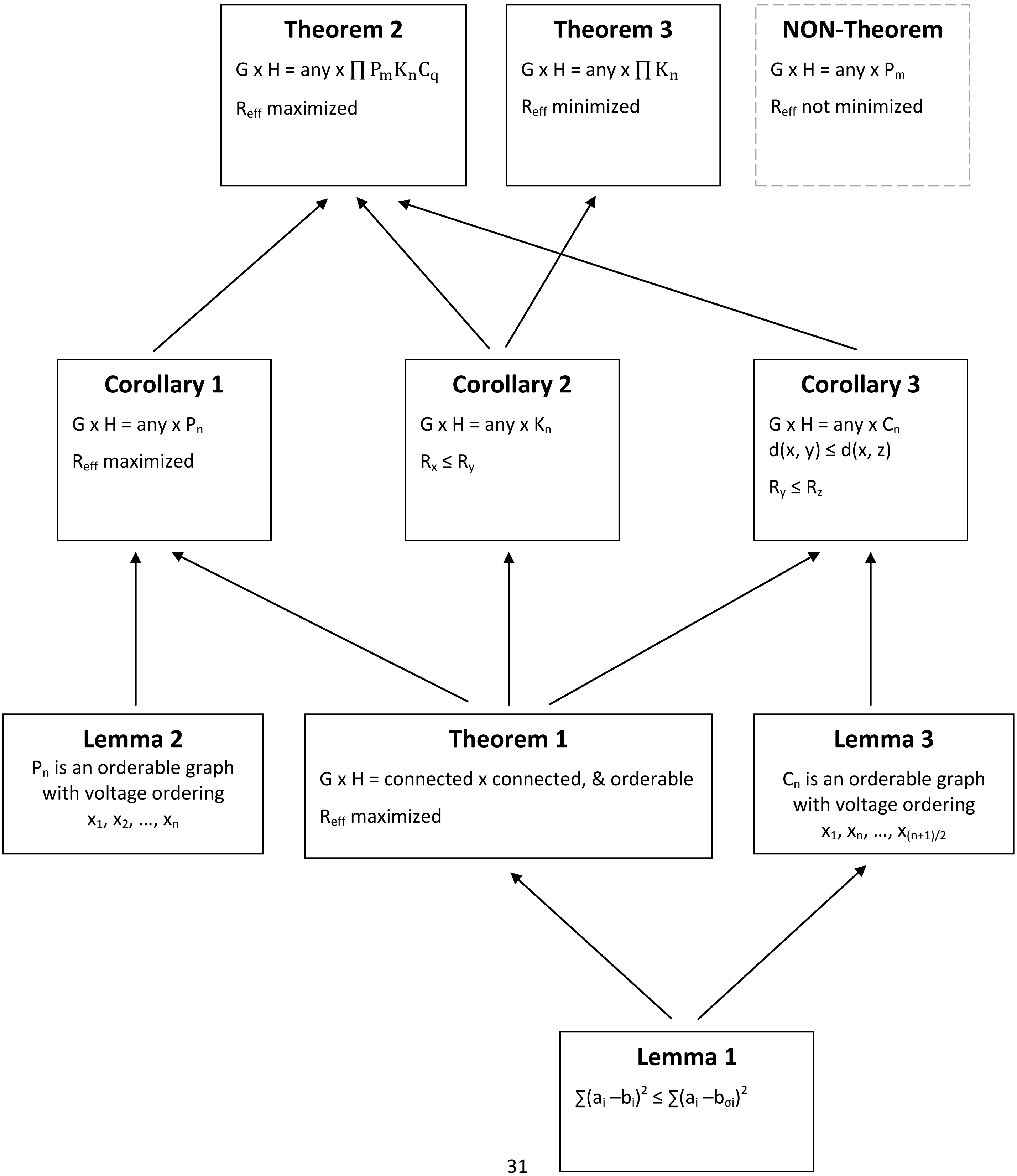
# Appendix

## A. Circuit Analysis for $P_3 \times P_3$

Below are the circuit analysis steps for determining the effective resistance between the two marked vertices.



**B. <u>Dependency Chart for Proofs of Results</u>**



## C. Alternative Method for Determining Effective Resistances in a Graph

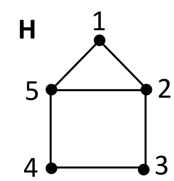
Effective resistances in a resistor network can be determined from an analysis of relationships between matrix representations of product graphs. First, we define some matrices that we will use to represent product graphs.

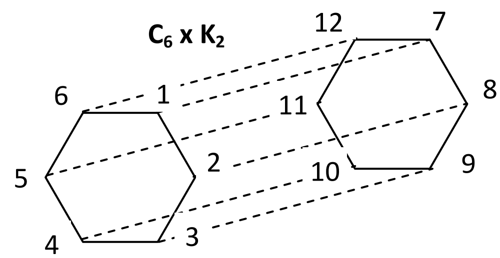
**1. Adjacency Matrix**

The adjacency matrix A of the graph G is the n x n matrix, where $a_{ij} = 1$ if there is an edge between vertex i and vertex j and $a_{ij} = 0$ otherwise. For example,

$A_H$ =

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 0 | 1 |
| 2 | 1 | 0 | 1 | 0 | 1 |
| 3 | 0 | 1 | 0 | 1 | 0 |
| 4 | 0 | 0 | 1 | 0 | 1 |
| 5 | 1 | 1 | 0 | 1 | 0 |




$A(C_6 \times K_2)$ =

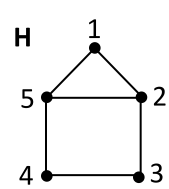
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 3 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 4 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 5 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 |
| 6 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 7 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| 8 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 |
| 9 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 |
| 10 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 |
| 11 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| 12 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 |

**2. Signed Edge-Vertex Adjacency Matrix**

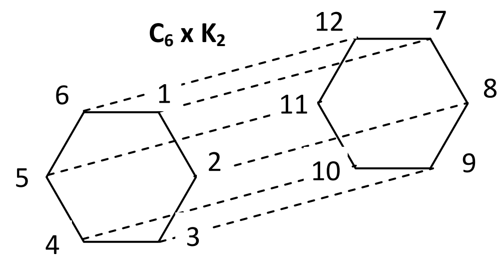
The signed edge-vertex adjacency matrix A of the graph G is the n x n matrix with rows indexed by edges and columns indexed by vertices, where $U_{(i,j), k} = 1$ if i = k, $U_{(i,j), k} = -1$ if j = k, and $U_{(i,j), k} = 0$ otherwise. For example,

$U_H$ =

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1, 2 | **1** | **-1** | 0 | 0 | 0 |
| 2, 3 | 0 | **1** | **-1** | 0 | 0 |
| 3, 4 | 0 | 0 | **1** | **-1** | 0 |
| 4, 5 | 0 | 0 | 0 | **1** | **-1** |
| 1, 5 | **1** | 0 | 0 | 0 | **-1** |
| 2, 5 | 0 | **1** | 0 | 0 | **-1** |



$U(C_6 \times K_2)$ =

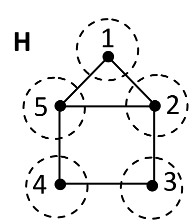


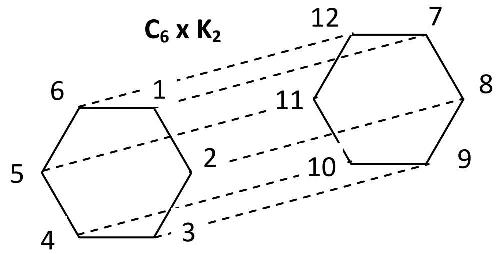

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1, 2 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2, 3 | 0 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3, 4 | 0 | 0 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4, 5 | 0 | 0 | 0 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5, 6 | 0 | 0 | 0 | 0 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6, 1 | 1 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1, 7 | -1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 2, 8 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 3, 9 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 4, 10 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 5, 11 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 6, 12 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 7, 8 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 1 | 0 | 0 | 0 | 0 |
| 8, 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 1 | 0 | 0 | 0 |
| 9, 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 1 | 0 | 0 |
| 10, 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 1 | 0 |
| 11, 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 1 |
| 12, 7 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | -1 |

**3. Degree Matrix**

The degree matrix D of the graph G is the n x n diagonal matrix, where $d_{ii} = \sum_{j=1}^{n} a_{ij}$. That is, each diagonal entry $d_{ii}$ is equal to the number of edges incident with vertex i. For example,

$$D_H = \begin{matrix} \mathbf{2} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{3} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{2} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{2} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{3} \end{matrix}$$






D($C_6$ x $K_2$) =

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | **3** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | **3** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | **3** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | **3** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | **3** | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | **3** | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | **3** | 0 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **3** | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **3** | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **3** | 0 | 0 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **3** | 0 |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **3** |

4. **Laplacian Matrix**

The Laplacian matrix L of the graph G is defined as $L = U^TU$. For example,

$L_H = U_H{}^T U_H$

$$= \begin{matrix} \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\ \mathbf{-1} & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ 0 & \mathbf{-1} & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{-1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} \end{matrix} \quad \mathrm{X} \quad \begin{matrix} \mathbf{1} & \mathbf{-1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{-1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{-1} & 0 \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{-1} \\ 0 & \mathbf{1} & 0 & 0 & \mathbf{-1} \end{matrix}$$

$$= \begin{matrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{matrix}$$

An equivalent alternative definition for the Laplacian matrix is L = D – A. Thus, it is an n x n matrix where

$L_{ij}$ = -1, if there is an edge between vertex i and vertex j,
$L_{ij}$ = degree of i, if i=j, and
$L_{ij}$ = 0, otherwise.

For example,

$$L_H = \begin{matrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{matrix} \quad - \quad \begin{matrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{matrix} \quad = \quad \begin{matrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{matrix}$$

For a proof of the equivalency of the two definitions, refer to lecture notes by Daniel A. Speilman [6] and **Error! Reference source not found.**.

$L(C_6 \times K_2)$ =

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | -1 | 0 | 0 | 0 | -1 | -1 | 0 | 0 | 0 | 0 | 0 |
| 2 | -1 | 3 | -1 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 |
| 3 | 0 | -1 | 3 | -1 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 |
| 4 | 0 | 0 | -1 | 3 | -1 | 0 | 0 | 0 | 0 | -1 | 0 | 0 |
| 5 | 0 | 0 | 0 | -1 | 3 | -1 | 0 | 0 | 0 | 0 | -1 | 0 |
| 6 | -1 | 0 | 0 | 0 | -1 | 3 | 0 | 0 | 0 | 0 | 0 | -1 |
| 7 | -1 | 0 | 0 | 0 | 0 | 0 | 3 | -1 | 0 | 0 | 0 | -1 |
| 8 | 0 | -1 | 0 | 0 | 0 | 0 | -1 | 3 | -1 | 0 | 0 | 0 |
| 9 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | -1 | 3 | -1 | 0 | 0 |
| 10 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | -1 | 3 | -1 | 0 |
| 11 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | -1 | 3 | -1 |
| 12 | 0 | 0 | 0 | 0 | 0 | -1 | -1 | 0 | 0 | 0 | -1 | 3 |

## 5. Relationships between Graph Theory and Resistor Networks

To relate electrical properties of a resistor network to properties of a graph, we can let each edge (a, b) in a graph represent a resistor of unit resistance and make the following definitions.

$i_{(a,b)}$ = the current flowing through the edge (a, b) from a to b

$\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ = the vector of voltages at each vertex (node)

**i** = the vector of currents through each edge (resistor)

$\mathbf{i}_{ext}(a)$ = the current entering the graph through vertex (node) a

Then, from Ohm's Law, V = IR, the current across a resistor R with unit resistance and terminals a and b is

$i_{(a,b)} = (v_b - v_a)/R = (v_b - v_a)/1 = v_b - v_a$, and

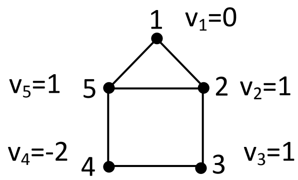
$\mathbf{i}_{ext}(a) = \sum i_{(a,b)}$ for all b such that (a,b) is in the edge set of the graph.

Also, we can write the vector **i** in matrix form as

$\mathbf{i} = \mathbf{U}\boldsymbol{v}$,

where U is the signed edge-vertex adjacency matrix.

For example, for the house graph, if $v = (0, 1, 1, -2, 1)^T$, then




$$\mathbf{i} = \mathbf{U}\boldsymbol{v} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix} \begin{matrix} 0 \\ 1 \\ 1 \\ -2 \\ 1 \end{matrix} = \begin{matrix} -1 \\ 0 \\ 3 \\ -3 \\ 1 \\ 2 \end{matrix}$$

$\mathbf{i}_{ext}(a) = \sum i_{(a,b)} = \sum(v_b - v_a) = U^Ti = U^TUv$

Since $L = U^TU$, we have

$\mathbf{i}_{ext}(a) = L\boldsymbol{v}$

Then, if L is invertible, we can solve for $\boldsymbol{v}$. If L is not invertible, we solve instead by multiplying both sides on the left by the pseudo-inverse of L, where the pseudo-inverse of a symmetric matrix is the inverse on the range of the matrix. For a matrix L with eigenvalues $\gamma_1, \ldots, \gamma_n$ and corresponding normalized eigenvectors $u_1, \ldots, u_n$, the pseudo-inverse, L+, is defined as

$L^+ = \sum_{i:\gamma i \neq 0}(1/\gamma i)u_iu_i^T$.

For example, for $L(C_6 \times K_2)$, one eigenvalue with corresponding normalized eigenvector is

$\gamma_1 = 4$; $u_1 = (0.29, -0.29, 0.29, -0.29, 0.29, -0.29, 0.29, -0.29, 0.29, -0.29, 0.29, -0.29)$,

so that

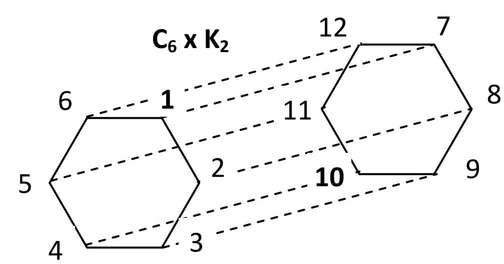
$(^1/_\gamma)_1(u_1 \cdot u_1^T) =$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 |
| -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 |
| 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 |
| -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 |
| 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 |
| -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 |
| 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 |
| -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 |
| 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 |
| -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 |
| 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 |
| -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 | -0.02 | 0.02 |

Adding these matrices for the nonzero eigenvectors, 4, 2, 6, 1, 1, 5, 5, 3, 3, 3, and 3, we have the pseudo-inverse matrix

$L^+ =$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.39 | 0.07 | -0.08 | -0.13 | -0.08 | 0.07 | 0.10 | 0.00 | -0.10 | -0.14 | -0.10 | 0.00 |
| 0.07 | 0.39 | 0.07 | -0.08 | -0.13 | -0.08 | 0.00 | 0.10 | 0.00 | -0.10 | -0.14 | -0.10 |
| -0.08 | 0.07 | 0.39 | 0.07 | -0.08 | -0.13 | -0.10 | 0.00 | 0.10 | 0.00 | -0.10 | -0.14 |
| -0.13 | -0.08 | 0.07 | 0.39 | 0.07 | -0.08 | -0.14 | -0.10 | 0.00 | 0.10 | 0.00 | -0.10 |
| -0.08 | -0.13 | -0.08 | 0.07 | 0.39 | 0.07 | -0.10 | -0.14 | -0.10 | 0.00 | 0.10 | 0.00 |
| 0.07 | -0.08 | -0.13 | -0.08 | 0.07 | 0.39 | 0.00 | -0.10 | -0.14 | -0.10 | 0.00 | 0.10 |
| 0.10 | 0.00 | -0.10 | -0.14 | -0.10 | 0.00 | 0.39 | 0.07 | -0.08 | -0.13 | -0.08 | 0.07 |
| 0.00 | 0.10 | 0.00 | -0.10 | -0.14 | -0.10 | 0.07 | 0.39 | 0.07 | -0.08 | -0.13 | -0.08 |
| -0.10 | 0.00 | 0.10 | 0.00 | -0.10 | -0.14 | -0.08 | 0.07 | 0.39 | 0.07 | -0.08 | -0.13 |
| -0.14 | -0.10 | 0.00 | 0.10 | 0.00 | -0.10 | -0.13 | -0.08 | 0.07 | 0.39 | 0.07 | -0.08 |
| -0.10 | -0.14 | -0.10 | 0.00 | 0.10 | 0.00 | -0.08 | -0.13 | -0.08 | 0.07 | 0.39 | 0.07 |
| 0.00 | -0.10 | -0.14 | -0.10 | 0.00 | 0.10 | 0.07 | -0.08 | -0.13 | -0.08 | 0.07 | 0.39 |




To determine the effective resistance R(1, 10), between vertices 1 and 10, we let $i_1 = -1$ and $i_{10} = 1$ so that

$i_{ext} = (-1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0)$.

Then,

$\boldsymbol{v} = L^+\mathbf{i}_{ext} = (-0.53, -0.17, 0.08, 0.23, 0.08, -0.17, -0.23, -0.08, 0.17, 0.53, 0.17, -0.08)$.

Thus, we have

$$R(1, 10) = \sum_{i=1}^{12} v_i^2$$

$$= (-0.53)^2 + (-0.17)^2 + 0.08^2 + 0.23^2 + 0.08^2 + (-0.17)^2 + (-0.23)^2 + (-0.08)^2 + 0.17^2 + 0.53^2 + 0.17^2 + (-0.08)^2$$

$$= 1.06.$$